Topology and its Applications 234 (2018) 335–347

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Real and Complex Singularities and their applications in Geometry and Topology

Whitney, Kuo–Verdier and Lipschitz stratifications for the surfaces $y^a = z^b x^c + x^d$

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A R T I C L E I N F O

Article history:
Received 7 January 2017
Received in revised form 24 July 2017
Accepted 10 August 2017
Available online 24 November 2017

MSC:
14J17
14P05
32S15

Keywords:
Algebraic surface
Singularity
Stratification
Whitney conditions
Kuo–Verdier regularity
Lipschitz stratification

A B S T R A C T

We specify the canonical stratifications satisfying respectively Whitney (a)-regularity, Whitney (b)-regularity, Kuo–Verdier (w)-regularity, and (L)-regularity for the family of surfaces $y^a = z^b x^c + x^d$, where a, b, c, d are positive integers, in both the real and complex cases.

1. Introduction

In this paper we give diagrams showing when the regularity conditions imposed by Whitney on stratifications known as (a) and (b), the condition (w) introduced by Kuo and Verdier, and the condition (L) of Mostowski, respectively hold for the stratification with two strata given by $(V - Oz, Oz)$ where V is the germ of the algebraic surface

$$\{(x, y, z) : y^a = z^b x^c + x^d\}$$

in $\mathbb{R}^3$ or $\mathbb{C}^3$ for positive integers a, b, c, d.

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https://doi.org/10.1016/j.topol.2017.11.030
0166-8641/

The third author was present at a workshop in Göttingen in October 1973, when Wall proposed that one make such a systematic study of the canonical (a)-regular and (b)-regular stratifications for this particular family of surfaces. The third author then began such a study in his Warwick thesis [25], finally completing the calculations required in [27].

Kuo has often said that it is important to understand the regularity properties of the stratification of this family, in particular for constructing examples illustrating general phenomena. He gave in 1971 the first such calculations [10] for the cases

$$(a, b, c, d) = (2k, 2, 2(k + \ell) + 1, 2(k + \ell + m) + 1),$$

with $k > 0, \ell \geq 0, m \geq 0$, in his paper introducing the ratio test (r) as a sufficient criterion for Whitney (b)-regularity for semi-analytic strata. (Kuo's proof that (r) implies (b) works also for sub-analytic strata [31], and in fact (r) implies (b) for definable stratifications in any o-minimal structure, as shown recently by Valette and the third author [29].)

A study of the family $\{(x, y, z) : y^a = z^b x^c + x^d\}$, for the (w)-regularity of Verdier, was undertaken by the second author in his thesis [18], completing earlier unpublished calculations by Kuo and the third author. Another systematic study was carried out by the first author in her thesis [4] for Mostowski's (L)-regularity. Both of these theses were directed by the third author at the University of Provence in Marseille.

Diagrams 1–4 in section 3 below were given by the third author in [27] in a publication of the University of Paris 7, with limited diffusion. Diagrams 5–6 below are from the second author's thesis [18]. Diagrams 7–8 below are from the first author's thesis ([4], [5]). It is clearly useful to have the results of these systematic calculations available, and in the same place. For example using the diagrams it becomes straightforward to specify infinitely many real algebraic examples where (b)-regularity holds but (w) fails. Previously only three such real algebraic examples had been found, in the 1979 paper of Brodersen and the third author [1]. These correspond to the choices $(a, b, c, d) \in \{(3, 2, 3, 5), (4, 4, 1, 3), (4, 2, 5, 7)\}$. Note that Verdier explicitly stated in early 1976 [31] that he knew of no sub-analytic example satisfying (b) but not (w). The first semi-algebraic example was constructed at Oslo in August 1976 by the third author [26].

The continued need for more available information about real algebraic examples, as provided here, is illustrated by a recent paper by Cluckers, Comte and Merle [2], where the authors mention that (b) does not imply (w) for real algebraic varieties, but give no reference. René Thom proposed around 1985 that more needed to be understood concerning the difference between (b) and (w).

Another area in which it is important to understand the difference between Whitney (b)-regularity and Kuo–Verdier (w)-regularity is the study of continuity of the density along strata of a definable regular stratification. For definable sets in arbitrary o-minimal structures (w)-regularity implies continuity of the density (in fact the density is a Lipschitz function as shown in [30] and [17]) while the density can fail to be continuous along a stratum of a (b)-regular stratification of a definable set if the o-minimal structure is not polynomially bounded [29].

Yet another difference was shown by Navarro Aznar and the third author for subanalytic stratifications: $(w) \Longrightarrow (w^*)$, but $(b) \Longrightarrow (b^*)$ is unknown except when the small stratum has dimension one [15]. We recall that the notion of $E^\star$-regularity, for E an equisingularity condition, came from the discussion of Teissier in his 1974 Arcata lectures [24]. Teissier stated there that one of the requirements for an equisingularity condition E to be 'good' is that it be preserved after intersection with generic linear spaces containing a given linear stratum and denoted this property by $E^\star$.

We observe that the algebraic surfaces $\{y^a = z^b x^c + x^d\}$ are quasi-homogeneous with weights $(ab, bd, a(d - c))$ and degree abd, when $c < d$, which is the situation where the different types of regularity are in doubt. In stratification theory, one often studies quasi-homogeneous spaces. Koike in [8] and [9], studied modified Nash triviality in the case of quasi-homogeneous families with isolated singularities. Fukui and

Paunescu [3] found some conditions on the equations defining the strata to show that a given family is topologically trivial. They gave a weighted version of (w)-regularity and of Kuo's ratio test, thus providing weaker sufficient conditions for topological stability. Sun [23] used these methods to generalize the work of the third author and Wilson [28] for (t)-regularity. The first author and Valette in [7] gave some conditions for a quasi-homogeneous stratification to satisfy (w)-regularity or to be bi-Lipschitz trivial (which means to satisfy the conclusion of the bi-Lipschitz Isotopy Lemma).

2. Definitions

The notion of stratification is one of the most fundamental concepts in algebraic geometry and singularity theory. The idea of decomposing a singular topological space X into non singular parts which we call the strata, satisfying some regularity conditions, goes back to Whitney and Thom's work. They suggested the use of stratifications as a method of understanding the geometric structure of singular analytic spaces.

At first, Whitney defined the stratification of a real algebraic set by iteratively taking the singular parts. The problem with this stratification is the absence of topological triviality along strata. Later Whitney introduced two famous regularity conditions for a stratification, called Whitney (a) and (b)-regularity [32].

These regularity conditions are defined in the following way:

Definition 2.1. Let X, Y be disjoint C^1 submanifolds of $\mathbb{R}^n$ such that $Y \cap \overline{X} \neq \emptyset$.

(i) The pair (X, Y) is said to be (a)-regular at a point $y \in Y \cap \overline{X}$ if for each sequence of points $\{x_i\}$ in X converging to y such that the sequence of tangent spaces $\{T_{x_i}X\}$ converges to α, then $T_yY \subset \alpha$.

(ii) The pair (X, Y) is said to be (b)-regular at a point $y \in Y \cap \overline{X}$ if for each pair of sequences of points $\{x_i\}$ in X and $\{y_i\}$ in Y converging to y such that the sequence of tangent spaces $\{T_{x_i}X\}$ converges to α, and the sequence of unit vectors $\overline{x_i y_i}$ converges to λ, then $\lambda \subset \alpha$.

Whitney showed that all analytic varieties in $\mathbb{C}^n$ have a stratification satisfying these conditions. Thom used the Whitney conditions in the differentiable case and proved that (b)-regular stratifications are topologically trivial along strata; this is called Thom's isotopy Theorem.

Another important condition is the (w)-regularity introduced by Kuo [10] and Verdier [31].

Definition 2.2. The pair (X, Y) as above is said to be (w)-regular at $y_0 \in Y \cap \overline{X}$ if there is a real constant $C > 0$ and a neighborhood U of y_0 such that

$$\delta(T_yY, T_xX) \leq C|x - y|$$

for all $y \in U \cap Y$ and all $x \in U \cap X$.

Here we set $\delta(A, B) = Sup\{\, a/||a||, b/||b|| \, : a \in A, b \in B^{\perp}\}$ for linear subspaces A, B of a Euclidean vector space.

Using Hironaka's resolution of singularities, Verdier established that every analytic variety, or subanalytic set, admits a (w)-regular stratification. He also showed that (w)-regularity of a subanalytic stratification implies (b)-regularity. This proof was then generalized by Tà Lê Loi to the general case of definable stratifications in an arbitrary o-minimal structure [11]. In general, for C^2 differentiable stratifications, (w)-regularity also implies local topological triviality along strata, see [31].

In his thesis [25], the third author gave the first proof that (b)-regularity is strictly weaker than (w)-regularity for both semi-algebraic stratified sets and real algebraic sets. But (w)-regularity is not strong enough to imply the stability of such characteristics of singularities as order of contact between pairs of arcs as one moves along a stratum. An example, due to Koike, was described in detail in [6]. This is

$$X = \{y^2 = z^2x^2 + x^3, x \geq 0\}.$$

The order of contact at the origin of the two branches of the curve $X_c = X \cap \{z = c\}$ for $c \neq 0$ is different from that of the two branches at the origin of X_0. But (w)-regularity holds along the z-axis.

The stronger the conditions imposed on the stratified set, the better the understanding of the geometry of this set. One of the strongest generic conditions is the (L)-regularity (defined below) introduced by Mostowski in 1985 [14]. Mostowski defined this notion of Lipschitz stratification and proved the existence of such stratifications for complex analytic sets. The existence of Lipschitz stratifications for real analytic sets, then for semi-analytic sets and subanalytic sets, was later proved by Parusiński in [19–22]. Recently Nguyen and Valette have proved the existence of Lipschitz stratifications of definable sets in all polynomially bounded o-minimal structures [16]. Lipschitz stratifications ensure local bi-Lipschitz triviality of the stratified set along each stratum, and bi-Lipschitz homeomorphisms preserve sets of measure zero, order of contact, and Łojasiewicz exponents. The condition (L) of Mostowski is preserved after intersection with generic wings, that is (L)-regularity implies (L^*)-regularity, see [6]; this was one of the criteria imposed on any good equisingularity notion by Teissier in his foundational 1974 Arcata paper [24]. We recall now the definition of Lipschitz stratification due to Mostowski, where we simplify slightly the (equivalent) formulation of the definition.

Let X be a closed subanalytic subset of an open subset of $\mathbb{R}^n$. By a *stratification* of X we shall mean a family $\Sigma = \{S^j\}_{j=l}^m$ of closed subanalytic subsets of X defining a filtration:

$$X = S^m \supset S^{m-1} \supset \cdots \supset S^l \neq \emptyset$$

and $\mathring{S}^j = S^j \setminus S^{j-1}$, for $j = l, l+1, \ldots, m$ (where $S^{l-1} = \emptyset$), is a smooth manifold of pure dimension j or empty. We call the connected components of $\mathring{S}^j$ the *strata* of Σ. We denote the function measuring distance to S^j by d_j, so that $d_j(q) = \mathrm{dist}(q, S^j)$. Set $d_{l-1} \equiv 1$, by convention (this will be used in the definitions below).

Definition 2.3. Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \mathring{S}^j$ is a (strictly) decreasing sequence of indices $j = j_1, j_2, \cdots, j_r = l$ such that each j_s for $s \geq 2$ is the largest integer less than j_{s-1} for which

$$d_{j_s-1}(q) \geq 2\gamma^2 d_{j_s}(q).$$

For each $j_s \in \{j_1, \ldots, j_r\}$ choose a point $q_{j_s} \in \mathring{S}^{j_s}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leq \gamma d_{j_s}(q)$.

If there is no confusion, we will call the sequence of points (q_{j_s}) a chain of q.

For $q \in \mathring{S}^j$, let $P_q : \mathbb{R}^n \to T_q\mathring{S}^j$ be the orthogonal projection to the tangent space $T_q\mathring{S}^j$ and let $P_q^{\perp} = I - P_q$ be the orthogonal projection onto the normal space $(T_q\mathring{S}^j)^{\perp}$.

A stratification $\Sigma = \{S^j\}_{j=l}^m$ of X is said to be a *Lipschitz stratification* if for some constant $C > 0$ and for every chain $q = q_{j_1}, q_{j_2}, \cdots, q_{j_r}$ and every k for $2 \leq k \leq r$,

$$|P_q^{\perp} P_{q_{j_2}} \cdots P_{q_{j_k}}| \le C|q - q_{j_2}|/d_{j_k-1}(q) \tag{L1}$$

and for each $q' \in \overset{\circ}{S}{}^{j_1}$ such that $|q - q'| \le (1/2\gamma)d_{j_1-1}(q)$,

$$|(P_q - P_{q'})P_{q_{j_2}} \cdots P_{q_{j_k}}| \le C|q - q'|/d_{j_k-1}(q) \tag{L2}$$

and

$$|P_q - P_{q'}| \le C|q - q'|/d_{j_1-1}(q) \tag{L3}$$

In the above definition of (L1), (L2) and (L3) we denote by $|\ |$ the usual norm for linear operators.

Mostowski and Parusiński showed that Lipschitz stratifications satisfy a bilipschitz version of the Verdier isotopy theorem.

It is easy to see that condition (L1) in the definition of Lipschitz stratification of Mostowski implies (w)-regularity and that in our case, where there are only two strata, it is actually equivalent to (w)-regularity.

3. The classifications

In this section we give diagrams showing when (E)-regularity, with E successively a, b, w and L, holds for the stratification with two strata given by $(V - Oz, Oz)$ where V is the germ of the algebraic surface

$$\{(x, y, z) : y^a = z^b x^c + x^d\}$$

in $\mathbb{R}^3$ or $\mathbb{C}^3$ for positive integers a, b, c, d.

3.1. Classification for Whitney (a)-regular and (b)-regular stratifications

The long and detailed calculations for these stratifications can be seen in chapter 8 of the third author's thesis [25]. He gave the outstanding calculations for these in [27].

Let $f(x, y, z) = -y^a + z^b x^c + x^d$, $V = f^{-1}(0)$, Y be the z-axis and let $X = V - Y$. Then Whitney (a)-regularity is verified at $(0, 0, z_0)$ in $\mathbb{R}^2 \times \mathbb{R}$ for (X, Y) when

$$\frac{\frac{\partial f}{\partial z}(x, y, z)}{|(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})|}$$

tends to 0 as (x, y, z) tends to $(0, 0, z_0)$ on X. A similar characterization is valid in $\mathbb{C}^3$.

For Whitney (b)-regularity, the third author used (b^π)-regularity. The pair (X, Y) is said to be (b^π)-regular at $(0, 0, z_0) \in Y \cap \overline{X}$ when every limit of tangent spaces to X at $(0, 0, z_0)$ defined by a sequence $\{p_i\}$ in X contains the limiting direction of secants $p_i\pi(p_i)$ where π is a (linear) retraction onto Y of a tube around Y defined near $(0, 0, z_0)$. When both (a) and (b^π) hold, X is (b)-regular. And for the calculation, the (b^π)-condition is verified at $(0, 0, z_0)$ in $\mathbb{R}^2 \times \mathbb{R}$ for (X, Y) when

$$\frac{x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial f}{\partial y}(x,y,z)}{|(x,y)|.|(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})|}$$

tends to 0 as (x,y,z) tends to $(0,0,z_0)$ in X. Again the analogous characterization is valid in $\mathbb{C}^3$.

It is easily verified using the Curve Selection Lemma for semi-algebraic sets that this condition holds if and only if the limits tend to 0 for (x,y,z) lying on an analytic curve $\alpha(u)=(x(u),y(u),z(u))$ such that $\alpha(0)=(0,0,z_0)$ and $\alpha(u)\in X$ if $u\neq 0$. The third author used this fact in some of the calculations required to complete the diagrams, but he used also a simpler approach by considering separately sequences (x_i,y_i,z_i) whose limiting direction is either tangent to $0\times\mathbb{R}$, tangent to $\mathbb{R}^2\times 0$ or neither of these, making three separate cases in all.

He distinguished between three distinct cases considering sequences (x_i,y_i,z_i) tending to $(0,0,0)$ with
(i) $\lim_{i\to\infty}\{\frac{x_i}{z_i}\}=0$.
(ii) $\lim_{i\to\infty}\{\frac{x_i}{z_i}\}=\kappa\neq 0$, $\kappa\in\mathbb{R}$.
(iii) $\lim_{i\to\infty}\{\frac{z_i}{x_i}\}=0$.

The difference between the diagrams in the real and complex cases arises solely from the non-existence of branches of the curve $\{z^b+x^{d-c}=0\}$ near 0 in $\mathbb{R}^2$ when b and $d-c$ are even and $d>c$. Such branches exist in $\mathbb{C}^2$, and they also exist for the curve $\{-z^b+x^{d-c}=0\}$ in $\mathbb{R}^2$.

3.2. Classification for Kuo–Verdier (w)-regular stratifications

It is known that w-regularity implies Whitney b-regularity [10], [31]. So we only need to study cases when b-regularity is satisfied in the classification obtained by the third author.

Again let $f(x,y,z)=-y^a+t^bx^c+x^d$ and let $v=(0,0,1)$. Then $\delta(Y,T_{(x,y,z)}X)=\frac{|grad\, f(x,y,z).v|}{||grad\, f(x,y,z)||}$, where we distinguish between the absolute value $|\ |$ defined on $\mathbb{R}$ and the norm $||\ ||$ defined on $\mathbb{R}^3$. Let π_2 be the projection defined by forgetting the second coordinate. For $(x,z)\in\pi_2(X)$, let $Z(x,z)=\frac{|grad\, f(x,y,z).v|}{||grad\, f(x,y,z)||.||(x,y)||}$ where y is such that $(x,y^a,z)\in\Gamma$ with Γ the graph of the smooth map $(x,z)\mapsto z^bx^c+x^d$. Then (w)-regularity is verified at 0 if and only if Z is bounded in the neighbourhood of 0 in $\pi_2(X)$. This implies that $Z(x,z)$ converges to 0 when $(x,z)\in\pi_2(X)$ converges to 0. The calculation uses the fact that Z is equivalent to Z', where:

$$Z'(x,z)=\frac{|z^{b-1}x^c|}{\sup(|cx^{c-1}z^b+dx^{d-1}|,|z^bx^c+x^d|^{\frac{a-1}{a}}).\sup(|x|,|z^bx^c+x^d|^{\frac{1}{a}})}$$

For the detailed calculations see the second author's thesis [N].

3.3. Classification for Lipschitz (L)-regular stratifications

Since the condition (L1) in the definition of Lipschitz stratification of Mostowski implies (w)-regularity, and in our case (where there only two strata) is actually equivalent, we need only to study cases when w-regularity is satisfied in the classification obtained by the second author [18]. And for these we need to verify the conditions (L2) and (L3) for $k=2$.

The essential technique in verifying (L2) and (L3) is to apply the mean value theorem to compare values at two points q and q' whose distance apart is controlled. Let $v=(0,0,1)$, $f(x,y,z)=-y^a+t^bx^c+x^d$, and write $\partial_q f$ for the gradient of f at a point q. To verify (L2), we must prove that:

$$\begin{aligned} |(P_q - P_{q'})P_{q_{j_2}}(v)| &= |(P_q - P_{q'})(0,0,1)| \\ &= |(P_q^{\perp} - P_{q'}^{\perp})(0,0,1) \\ &= \left| \frac{bx^c\, z^{b-1}\partial_q f}{||\partial_q f||^2} - \frac{bx'^c\, z'^{b-1}\partial_{q'} f}{|||\partial_{q'} f||^2} \right| \\ &\leq C\,|q-q'| \end{aligned}$$

To verify the condition (L3), we use the vector basis $v_1 = (1,0,0), v_2 = (0,1,0)$ and $v_3 = (0,0,1)$. Let $\varphi(x,y,z) = y - (z^b x^c + x^d)^{1/a}$. We prove that:

$$\begin{aligned} |(P_q - P_{q'})(v_1)| &= \left| \frac{\frac{\partial\varphi}{\partial x}(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})}{||\partial_q\varphi||^2} - \frac{\frac{\partial\varphi}{\partial x'}(\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})}{||\partial_{q'}\varphi||^2} \right| \\ &\leq C\,\frac{|q-q'|}{d(q,O_z)}, \end{aligned}$$

$$\begin{aligned} |(P_q - P_{q'})(v_2)| &= \left| \frac{\frac{\partial\varphi}{\partial y}(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})}{||\partial_q\varphi||^2} - \frac{\frac{\partial\varphi}{\partial y'}(\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})}{||\partial_{q'}\varphi||^2} \right| \\ &\leq C\,\frac{|q-q'|}{d(q,O_z)}, \end{aligned}$$

and

$$\begin{aligned} |(P_q - P_{q'})(v_3)| &= \left| \frac{\frac{\partial\varphi}{\partial z}(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})}{||\partial_q\varphi||^2} - \frac{\frac{\partial\varphi}{\partial z'}(\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})}{||\partial_{q'}\varphi||^2} \right| \\ &\leq C\frac{|q-q'|}{d(q,O_z)}. \end{aligned}$$

To check that the condition (L) fails, we give the two curves of q and q' where the condition (L2) or (L3) fails. Long and detailed calculations deciding several branches of the Diagram 8 (see below) were given in the first author's thesis [4].

4. The diagrams of the classification for Whitney, Kuo–Verdier and Lipschitz stratifications

Note: $\surd$means that regularity holds

$\times$ means that regularity fails

? means undecided

$b \equiv 0(2)$ means that b is even

$b \equiv 1(2)$means that b is odd, etc.

$$
\begin{cases}
a = 1 \quad \surd \\[2ex]
a > 1 \begin{cases}
d \le c \quad \surd \\[2ex]
d > c \begin{cases}
d \ge b + c \quad \times \\[2ex]
d < b + c \begin{cases}
a \le b \quad \surd \\[2ex]
a > b \begin{cases}
d < \frac{ac}{a-b} \quad \surd \\[2ex]
d \ge \frac{ac}{a-b} \quad \times
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
$$

Diagram 1. Whitney (a)-regularity in $\mathbb{C}^3$.

$$
\begin{cases}
a = 1 \quad \surd \\[2ex]
a > 1 \begin{cases}
d \le c \quad \surd \\[2ex]
c < d < b + c \begin{cases}
a \le b \quad \surd \\[2ex]
b < a \begin{cases}
d < \frac{ac}{a-b} \quad \surd \\[2ex]
d \ge \frac{ac}{a-b} \begin{cases}
b \equiv 0(2) \begin{cases}
d \equiv c(2) \quad \surd \\[2ex]
d \equiv c + 1(2) \quad \times
\end{cases} \\[2ex]
b \equiv 1(2) \quad \times
\end{cases}
\end{cases}
\end{cases} \\[2ex]
b + c \le d \begin{cases}
b \equiv 1(2) \quad \times \\[2ex]
b \equiv 0(2) \begin{cases}
d \equiv c + 1(2) \quad \times \\[2ex]
d \equiv c(2) \begin{cases}
a \le b \quad \surd \\[2ex]
b < a < b + c \begin{cases}
d < \frac{ac}{a-b} \quad \surd \\[2ex]
d \ge \frac{ac}{a-b} \quad \times
\end{cases} \\[2ex]
b + c \le a \quad \times
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
$$

Diagram 2. Whitney (a)-regularity in $\mathbb{R}^3$.

$$\begin{cases} a = 1 \ \ \surd \\ a > 1 \begin{cases} d \leq c \ \ \surd \\ d > c \ \ \times \end{cases} \end{cases}$$

Diagram 3. Whitney (b)-regularity in $\mathbb{C}^3$.

$$\begin{cases} a = 1 \ \ \surd \\ a > 1 \begin{cases} d \leq c \ \ \surd \\ d > c \begin{cases} b \equiv 1(2) \ \ \times \\ b \equiv 0(2) \begin{cases} d \equiv c + 1(2) \ \ \times \\ d \equiv c(2) \begin{cases} d < a \begin{cases} d < b + c \ \ \surd \\ d \geq b + c \ \ \times \end{cases} \\ a \leq d \begin{cases} a > c \ \ \times \\ a \leq c \begin{cases} d < b + c \ \ \times \\ d \geq b + c \begin{cases} a \leq b \ \ \surd \\ a > b \begin{cases} d < \frac{ac}{a-b} \ \ \surd \\ d \geq \frac{ac}{a-b} \ \ \times \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

Diagram 4. Whitney (b)-regularity in $\mathbb{R}^3$.

$$\begin{cases} a = 1 \ \ \surd \\ a > 1 \begin{cases} d \leq c \ \ \surd \\ d > c \ \ \times \end{cases} \end{cases}$$

Diagram 5. Kuo–Verdier (w)-regularity in $\mathbb{C}^3$.

$$\begin{cases} a=1 \quad \surd \\ a>1 \begin{cases} d\le c \quad \surd \\ d>c \begin{cases} b\equiv 1(2) \quad \times \\ b\equiv 0(2) \begin{cases} d\equiv c+1(2) \quad \times \\ d\equiv c(2)\begin{cases} a(d-c)\le b|d-a| \quad \surd \\ a(d-c)>b|d-a| \quad \times \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

Diagram 6. Kuo–Verdier (w)-regularity in $\mathbb{R}^3$.

$$\begin{cases} a=1 \quad \surd \\ a>1 \begin{cases} d\le c \quad \surd \\ d>c \quad \times \end{cases} \end{cases}$$

Diagram 7. Mostowski (L)-regularity in $\mathbb{C}^3$.

$$\begin{cases} a=1 \quad \surd \\ a>1 \begin{cases} d\le c \quad \surd \\ d>c \begin{cases} b\equiv 1(2) \quad \times \\ b\equiv 0(2) \begin{cases} d\equiv c+1(2) \quad \times \\ d\equiv c(2)\begin{cases} a(d-c)\le b|d-a| \begin{cases} d\ge a \quad \text{(continued below)} \\ d<a \begin{cases} d,c \ \text{ even} \quad \times \\ d,c \ \text{ odd} \begin{cases} b<2(d-c) \quad \times \\ b\ge 2(d-c) \quad ? \end{cases} \end{cases} \end{cases} \\ a(d-c)>b|d-a| \quad \times \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

$$\to d \geq a \begin{cases} a \ \text{even} \ \times \\ a \ \text{odd} \begin{cases} c < a \ \times \\ c \geq a \begin{cases} b < a \begin{cases} a(d-c) \leq d-a \ \surd \\ a(d-c) > d-a \begin{cases} 2a(d-c) > db \ \times \\ 2a(d-c) \leq db \ ? \end{cases} \end{cases} \\ b \geq a \begin{cases} b \geq 2a \ \surd \\ b < 2a \begin{cases} a(d-c) \leq d \ \surd \\ a(d-c) > d \begin{cases} 2a(d-c) > db \ \times \\ 2a(d-c) \leq db \ ? \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

Diagram 8. Mostowski (L)-regularity in $\mathbb{R}^3$.

5. Some consequences

An immediate consequence of Diagram 1, determining the canonical (a)-regular stratification of V in $\mathbb{C}^3$ is:

Proposition. *Let $F : \mathbb{C}^2 \times \mathbb{C}^k, 0 \times \mathbb{C}^k \to \mathbb{C}, 0$ be a polynomial map whose singular set is $0 \times \mathbb{C}^k$, and let $F_z(x, y) = F(x, y, z)$, $k \geq 1$. Then there is no local analytic invariant α of plane curves in $\mathbb{C}^2$ such that $\alpha(F_z)$ is constant if and only if $(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k)$ is (a)-regular.*

Proof. If there were such an invariant α then because $\{y^a = z^b x^c + x^d\}$ is (a)-regular over the z-axis when $b \geq a > 1$ and $b > d - c > 0$ by Diagram 1, it would follow that $\alpha(-y^a + x^d) = \alpha(-y^a + \epsilon x^c + x^d)$ for $\epsilon \neq 0$. But this would imply that $\{y^a = z^b x^c + x^d\}$ is (a)-regular over the z-axis for all b, which is not the case by Diagram 1, in particular when $a > 1, b = 1$ and $d > c$. $\square$

Notes. 1. The Milnor number is a local analytic invariant whose constance is equivalent to (b)-regularity. Diagram 3 confirms this: the validity of (b)-regularity is independent of the power b of z in the deformation $-y^a + z^b x^c + x^d$.

2. By an example of Zariski [33], $\{y^2 = zwx^2 + x^3\}$, it was already known that there is no semicontinuous invariant α, because the set of values in the (z, w)-plane for which (a)-regularity fails in this example is $\{zw = 0\} - (0, 0)$, in particular this set is not closed [27]. But there do exist nonsemicontinuous numerical invariants of plane curves (cf. chapter 10 of Milnor's book [13]), so the proposition above provides new information.

3. When $b = 1$, there is an (a)-fault at 0 if and only if there is a (b)-fault at 0. This follows from Diagrams 1–4 and is rather surprising.

C^1-smooth varieties

A real algebraic variety can be C^1-smooth and yet be singular. In fact for all finite k there is a singular algebraic variety which is C^k-smooth. However Mather has proved [12] that if a variety is C^∞-smooth, then it is nonsingular. Obviously the converse is true too. The relevance to our study follows from the C^1-invariance of (a)- and (b)-regularity, which implies that both (a) and (b) hold if V is C^1-smooth. The following proposition, whose proof is an easy exercise, may be compared with Diagrams 2 and 4. The proposition applies when $d > c$, but when $d \leq c$ we have already seen that (a) and (b) hold.

Proposition. *Let $y^a = z^b x^c + x^d$ and let $d > c$. Then y is a C^1 function of x and z if and only if $a \leq c, d < ac/(a-b)$, b is even, and $(d-c)$ is even.*

When $g(x,z) = (z^b x^c + x^d)^{1/a}$ is a C^1 function of x and z, the graph of g is a C^1 submanifold, so that (b)-regularity holds.

Examples of (b)-regular (w)-faults

As noted earlier [1] provides three examples among the surfaces $y^a = z^b x^c + x^d$ where (b)-regularity holds but (w)-regularity fails:

$$(a,b,c,d) = (3,2,3,5), (4,2,5,7) \text{ or } (4,4,1,3).$$

The diagrams of the present paper furnish infinitely many such examples, as by Diagram 6 (w) fails when $a > 1, d > c, b \equiv 0(2), d \equiv c(2)$, and $a(d-c) > b|d-a|$, and by Diagram 4 (b) holds if further $a < c, d \geq b+c$ and either $a < b$ (case 1) or $a \geq b$ and $d(a-b) < ac$ (case 2). The examples $(3,2,3,5)$ and $(4,2,5,7)$ correspond to case 1, while $(4,4,1,3)$ corresponds to case 2.

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